Linear Mappings

- Linear mappings
- Eigenvalues and Eigenvectors.

Linear Mappings

- Linear mappings are functions defined on vector spaces that preserve linear combinations.

T : mapping, transformation.

Definition

Given two vector spaces V and W, we say that $T: V \to W$ is a linear mapping if it verifies: $\forall \bar{u}, \bar{v} \in V$ and $\forall \alpha \in \mathbb{R}$

(a) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ $\qquad \bar{u}+\bar{v} \in V$

(b) $T(\alpha \bar{u}) = \alpha T(\bar{u})$. $\qquad T(\bar{u}), T(\bar{v}) \in W$

Example. The mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} y \\ x \end{pmatrix}$ is linear.

$$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad T(\bar{u}) = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}, \quad T(\bar{v}) = \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}$$

$$T\left(\alpha \bar{u} + \beta \bar{v}\right) = \underbrace{\cdots\cdots}_{} = \alpha T(\bar{u}) + \beta T(\bar{v})$$

$$T(\alpha \bar{u} + \beta \bar{v}) = T\left[\begin{pmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \end{pmatrix}\right] = \begin{pmatrix} \alpha u_2 + \beta v_2 \\ \alpha u_1 + \beta v_1 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} + \beta \begin{pmatrix} v_2 \\ v_2 \end{pmatrix} = \alpha T(\bar{u}) + \beta T(\bar{v})$$

Example. The mapping $D : \mathbb{P} \to \mathbb{P}$ defined by $Dp(x) = p'(x)$ is linear.

$p, q \in \mathbb{P}$, $\alpha, \beta \in \mathbb{R}$

$$D[\alpha p + \beta q] = (\alpha p + \beta q)' \overset{\lim \ldots}{=} \alpha p' + \beta q'$$

$$= \alpha Dp + \beta Dq$$

Example. Let A be a matrix of size $m\times n$. The mapping $T:\mathbb{R}^n\to\mathbb{R}^m$ defined by $T(\bar{x})=A\bar{x}$ is linear.

$$\left.\begin{aligned} A(\bar{x}+\bar{y}) &= A\bar{x}+A\bar{y} \\ A(\alpha\bar{x}) &= \alpha A\bar{x} \end{aligned}\right\} \quad \begin{aligned} &A(\alpha\bar{x}+\beta\bar{y}) \\ &=\alpha A\bar{x}+\beta A\bar{y} \end{aligned}$$

$$\underbrace{T\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]=\begin{pmatrix} y \\ x \end{pmatrix}}_{T(\bar{x})=A\bar{x}}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

- Using the definition
- Showing that it can be written as $A\bar{x}$

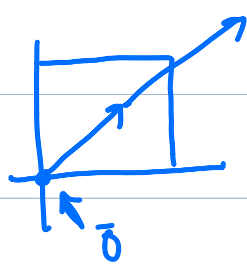
Example. The mapping $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f\left[\begin{pmatrix}x\\y\end{pmatrix}\right]=\begin{pmatrix}y\\x^2\end{pmatrix}$ is not linear.

- We can show that it does not preserve scalar multiplication.

- $f\left[5\begin{pmatrix}2\\1\end{pmatrix}\right]=f\left[\begin{pmatrix}10\\5\end{pmatrix}\right]=\begin{pmatrix}5\\100\end{pmatrix}=5\begin{pmatrix}1\\20\end{pmatrix}$

- $5f\left[\begin{pmatrix}2\\1\end{pmatrix}\right]=5\begin{pmatrix}1\\4\end{pmatrix}$

$$f\left[5\begin{pmatrix}2\\1\end{pmatrix}\right]\neq 5f\left[\begin{pmatrix}2\\1\end{pmatrix}\right]$$



V c. l. c

1. $\bar{0}\in V$
2. V c.a
3. V c.s.m

T p.l.c. $\Leftrightarrow$ T is linear.

1. $T(\bar{0}_V)=\bar{0}_W$
2. T p.a
3. T p.s.m

Properties of linear mappings

Let $T: V \to W$ be a linear mapping, then:

1. $T(\bar{0}_V) = \bar{0}_W$
2. $T(-\bar{u}) = -T(\bar{u})$ $\qquad -\bar{u} = -1 \cdot \bar{u}$
3. $T(a_1\bar{u}_1 + a_2\bar{u}_2 + \cdots + a_n\bar{u}_n) = a_1T(\bar{u}_1) + \cdots + a_nT(\bar{u}_n)$

$T(\alpha\bar{u} + \beta\bar{v})$

Example. The mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} x+1 \\ y \end{pmatrix}$ is not linear since $f\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$f\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\underbrace{f(\bar{x}) = A\bar{x}}_{\text{linear}} + \bar{b}$

Kernel and Image of a linear mapping

Definition

Let $T: V \to W$ be a linear mapping.
We define the kernel and image of T, respectively, as follows:

$$\operatorname{Ker} T = \{ \bar{x} \in V,\ T(\bar{x}) = \bar{0} \} \subseteq V$$

$$\operatorname{Im} T = \{ T(\bar{x}) \in W,\ \bar{x} \in V \} \subseteq W$$

$$= \{ \bar{y} \in W,\ \bar{y} = T(\bar{x}) \text{ for some } \bar{x} \in V \}$$

- $\operatorname{Ker} T$ is a subspace of V
- $\operatorname{Im} T$ is a subspace of W.

$\operatorname{Ker} T$ is a subspace:

$\bar{x}, \bar{y} \in \operatorname{Ker} T \quad \rightsquigarrow \quad \alpha \bar{x} + \beta \bar{y} \in \operatorname{Ker} T$

- $T(\bar{x}) = \bar{0}$
- $T(\bar{y}) = \bar{0}$

$\to \quad T(\alpha \bar{x} + \beta \bar{y}) = \alpha T(\bar{x}) + \beta T(\bar{y})$

$= \alpha \bar{0} + \beta \bar{0} = \bar{0}$

Im T is a subspace.

$\bar{u}, \bar{v} \in \operatorname{Im} T \xrightarrow{T \text{ is linear}} \alpha\bar{u} + \beta\bar{v} \in \operatorname{Im} T$

def of Im T

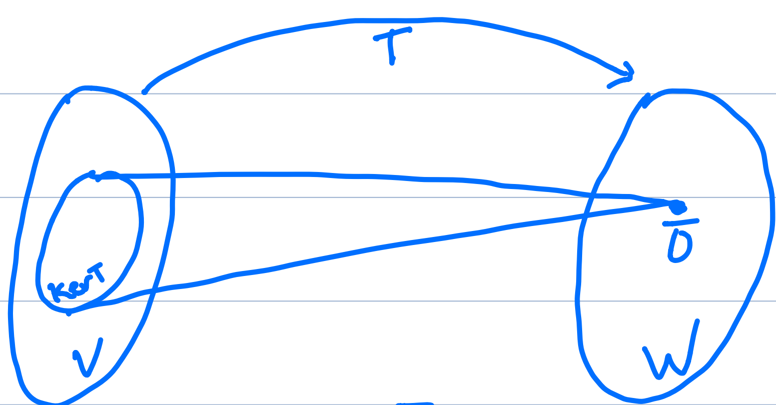
$\bar{x}, \bar{y} \in V \quad \bar{u} = T(\bar{x})$

$\bar{v} = T(\bar{y})$

def of Im T

$\exists\, \bar{z} \in V, \quad \underline{\alpha\bar{u} + \beta\bar{v}} = T(\underline{\bar{z}})$

$$\alpha\bar{u} + \beta\bar{v} = \alpha T(\bar{x}) + \beta T(\bar{y})$$
$$= T(\underbrace{\alpha\bar{x} + \beta\bar{y}}_{\bar{z}})$$

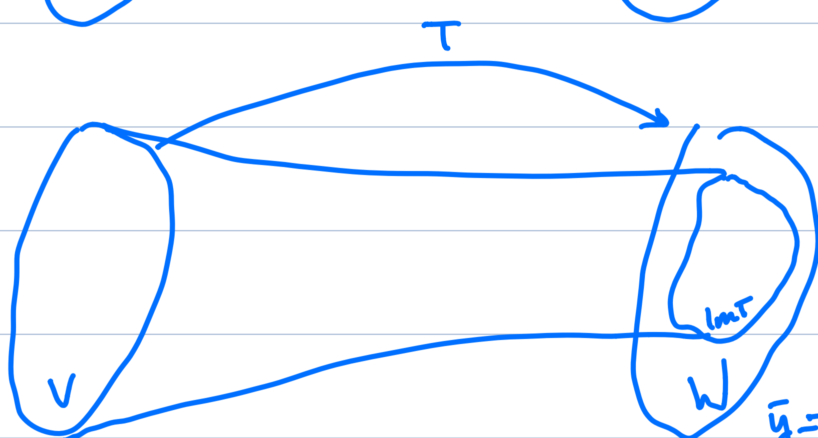


• $T(\bar{x}) = A\bar{x}$

$V = W$

$V = \mathbb{R}^p \quad W = \mathbb{R}^n$

$\operatorname{Ker} T = \operatorname{Nul} A$



$\operatorname{Im} T \subseteq W$

$\operatorname{Im} T = \operatorname{Col} A$

$\underline{\bar{y} = A\bar{x}}$

T is surjective $\Leftrightarrow \operatorname{Im} T = W$

Theorem

Let $T: V \to W$ be a linear mapping and let $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n\}$ be a system of generators of V. Then $\{T(\bar{u}_1), T(\bar{u}_2), \dots, T(\bar{u}_n)\}$ is a system of generators of $\operatorname{Im} T$.

$$V = \operatorname{Span}\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n\} \longrightarrow \operatorname{Im} T = \operatorname{Span}\{T(\bar{u}_1), T(\bar{u}_2), \dots, T(\bar{u}_n)\}$$

Example. Find the kernel and image of the linear mapping $f: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} x+z \\ y \\ x+2y+z \end{pmatrix}$$

$T(\bar{x}) = A\bar{x}$

$\to \ker f = \operatorname{Nul} A$

$\to \operatorname{Im} f = \operatorname{Col} A$

$T: V \to W$

$$f\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x + z &= 0 \\ y &= 0 \\ z &= \lambda \end{aligned}$$

$$\mathrm{Nul}\, A = \mathrm{Ker}\, f = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\boxed{\mathrm{Col}\, A = \mathrm{Im}\, f} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathrm{Span}\{f(\bar{u}_1), f(\bar{u}_2), f(\bar{u}_3)\}$$

$$= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

If we apply the previous theorem

$$\bar{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} x+z \\ y \\ x+2y+z \end{pmatrix}$$

$$f(\bar{u}_1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad f(\bar{u}_2) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad f(\bar{u}_3) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\operatorname{Im} T = \{ T(\bar{x}) : \bar{x} \in V \} \overset{T(\bar{x}) = A\bar{x}}{=} \{ A\bar{x} : \bar{x} \in V \}$$

$$= \{ x_1\bar{a}_1 + x_2\bar{a}_2 + \cdots + x_n\bar{a}_n : \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in V$$

$$= \operatorname{Col} A$$

$$\operatorname{Ker} T = \{ \bar{x} \in V : T(\bar{x}) = \bar{0} \} \overset{T(\bar{x}) = A\bar{x}}{=} \{ \bar{x} \in V : A\bar{x} = \bar{0} \}$$

$$= \operatorname{Nul} A.$$

Injective, surjective, and bijective mapping

- A function $f: A \to B$ is injective if and only if

  $\forall x, y \in A$, if $x \neq y$ then $f(x) \neq f(y)$,

  or equivalently

  $\forall x, y \in A$, if $f(x) = f(y)$ then $x = y$.

f is injective iif $f(x)=y$ has at most 1 sol. $\forall \bar{y}$

- A function $f: A \to B$ is surjective if and only if

  $\forall b \in B$, there exists $a \in A$ such that

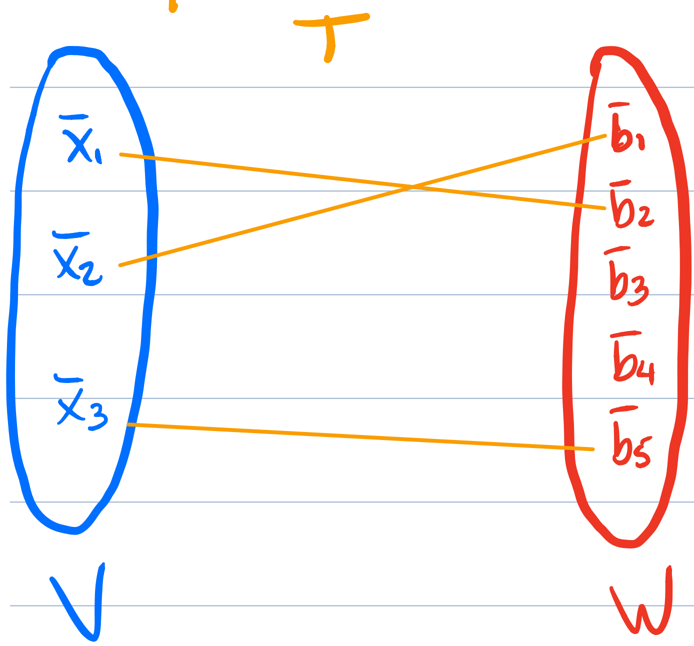
  $$b = f(a).$$

f is surjective iff $f(\bar{x}) = \bar{y}$ has at least 1 sol. $\forall \bar{y}$

- A function is bijective if and only if it is both injective and surjective.

f is bijective iff $f(\bar{x}) = \bar{y}$ always has a unique solution.

Example.



$\bar{b}_1 = T(\bar{x}) \rightarrow \bar{x} = \bar{x}_2$

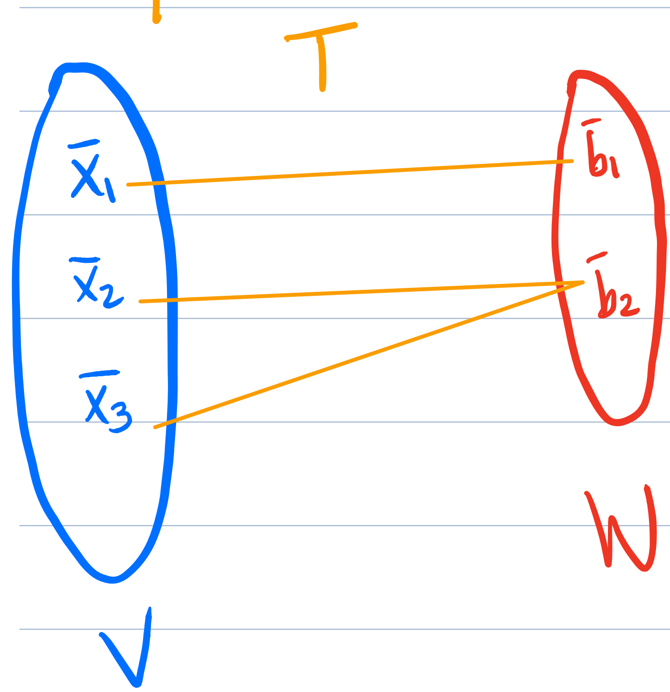
$\bar{b}_2 = T(\bar{x}) \rightarrow \bar{x} = \bar{x}_1$

$\bar{b}_5 = T(\bar{x}) \rightarrow \bar{x} = \bar{x}_3$

Injective.

Not surjective.

Example.

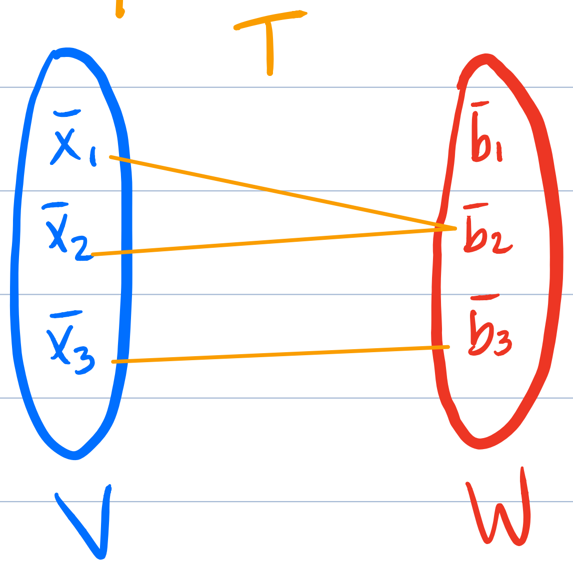


$\bar{b}_2 = T(\bar{x})$ $\rightarrow \bar{x} = \bar{x}_2$, $\rightarrow \bar{x} = \bar{x}_3$

Not injective.

Surjective.

Example.

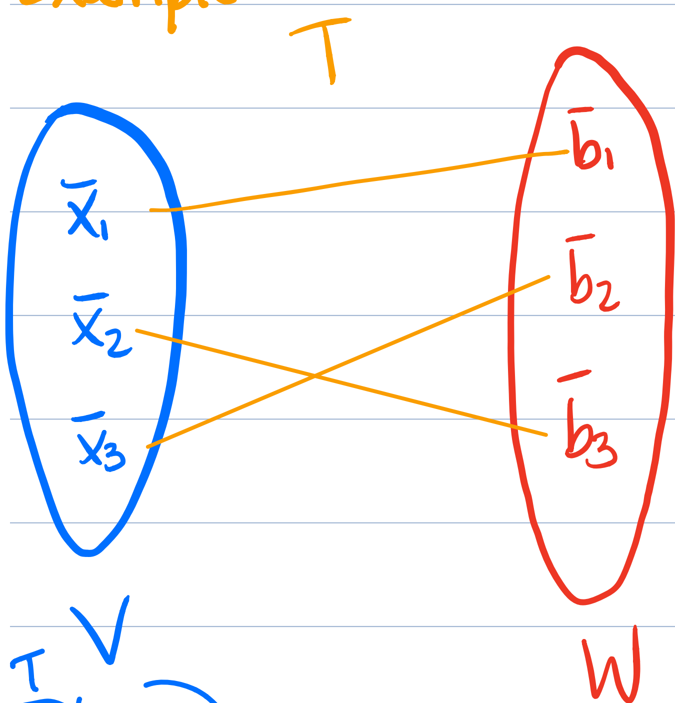


$$\bar{b}_2 = T(\bar{x}) \begin{cases} \bar{x} = \bar{x}_1 \\ \bar{x} = \bar{x}_2 \end{cases}$$

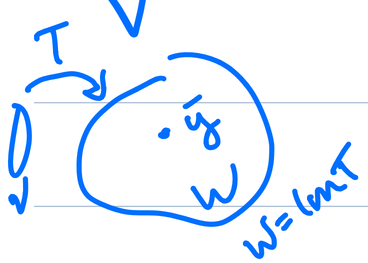
Not injective

Not surjective.

Example.



$\left.\begin{array}{l}\text{Injective.} \\ \text{Surjective.}\end{array}\right\}$ bijective.



$T(\bar{x}) = \bar{y}$

$T(\bar{x}) = A\bar{x} = \bar{y}$

$\boxed{\text{Theorem}}$

Let $T: V \to W$ be a linear mapping. Then:

1. T is injective if and only if $\ker T = \{\bar{0}\}$.
2. T is surjective if and only if $\operatorname{Im} T = W$.

Other characterizations

For any given linear mapping $f: V \to W$, we have:

1. f is injective if and only if for each linearly independent set $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n\}$, the set $\{f(\bar{u}_1), f(\bar{u}_2), \dots, f(\bar{u}_n)\}$ is linearly independent.

f is injective $\iff$ f preserves lin. independence.

$\Rightarrow$ Suppose that f is injective.

NTS: if $\{\bar{u}_1, \dots, \bar{u}_n\}$ is lin. ind.

then $\{f(\bar{u}_1), \dots, f(\bar{u}_n)\}$ lin. ind.

$$c_1 f(\bar{u}_1) + c_2 f(\bar{u}_2) + \dots + c_n f(\bar{u}_n) = \bar{0}_W$$

$$f(\underbrace{c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_n\bar{u}_n}_{\bar{0}_V}) = \bar{0}_W$$

f is injective $\Rightarrow$ $c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_n\bar{u}_n = \bar{0}_V$

$\{\bar{u}_1, \dots, \bar{u}_n\}$ is lin ind $\Rightarrow c_1 = c_2 = c_3 = \dots c_n = 0$

$\Leftarrow$ Suppose if $\{f(\bar{u}_1), \dots, f(\bar{u}_n)\}$ is lin. dep.

then $\{\bar{u}_1, \dots, \bar{u}_n\}$ is lin dep.

NTS: f is injective. $\Leftrightarrow$ $\ker f = \{\bar{0}\}$

$\bar{x} \in \ker f$ $\qquad f(\bar{x}) = \bar{0}$

lin. dep $\{\bar{0}\} = \{f(\bar{x})\} \Rightarrow \{\bar{x}\}$ is l.d

$\bar{x} = \bar{0}$

2. f is surjective if and only if for each system of generators of V $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n\}$, the set $\{f(\bar{u}_1), f(\bar{u}_2), \dots, f(\bar{u}_n)\}$ is a system of generators of W.

3. f is bijective if and only if for each basis of V $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m\}$, the set $\{f(\bar{u}_1), f(\bar{u}_2), \dots, f(\bar{u}_m)\}$ is a basis of W.

Operations with linear mappings

Given two linear mappings $f, g: V \to W$ and a scalar $\lambda \in \mathbb{R}$, we define the operations

$$f+g: V \to W; \quad (f+g)(\bar{x}) = f(\bar{x}) + g(\bar{x})$$

$$\lambda f: V \to W; \quad (\lambda f)(\bar{x}) = \lambda f(\bar{x})$$

V, W $\{T \text{ linear mapping}: T: V \to W\}$

Theorem

$\boxed{\text{With the operations defined above, the set of linear mappings between two vector spaces } V \text{ and } W \text{ is itself a vector space.}}$

Moreover, the set of linear mappings between V and W has additional structure:

$T(\bar{x}) = \bar{0}$

$T(\bar{v}) = 0\bar{x}$

Theorem

Given two linear mappings $f: V \to W$ and $g: W \to U$, their composition $g \circ f: V \to U$ defined by $(g \circ f)(\bar{x}) = g(f(\bar{x}))$ is also a linear mapping.

Recall that if a function is bijective then it has an inverse function.

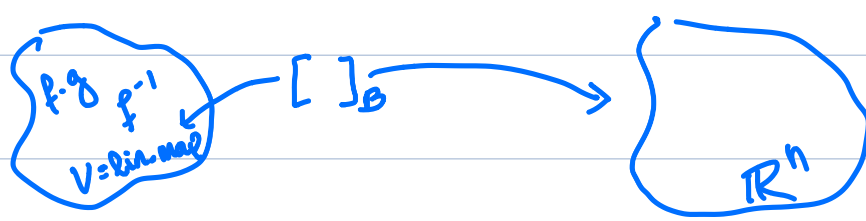
In particular, if $f: V \to W$ is a bijective linear mapping, then there exists a function $f^{-1}: W \to V$ such that

$$\forall \bar{v} \in V, \quad f^{-1}(f(\bar{v})) = \bar{v} = Id_V \qquad V \to W \to V$$

$$\forall \bar{w} \in W, \quad f(f^{-1}(\bar{w})) = \underset{\underset{Id_W}{\shortparallel}}{\bar{w}}. \qquad W \to V \to W$$

Theorem

> For each bijective linear function $f: V \to W$, its inverse mapping $f^{-1}: W \to V$ is also a linear mapping. That is,
>
> $$f^{-1}(\alpha \bar{u} + \beta \bar{v}) = \alpha f^{-1}(\bar{u}) + \beta f^{-1}(\bar{v}), \quad \forall \bar{u}, \bar{v} \in W$$
>
> $$\forall \alpha, \beta \in \mathbb{R}.$$



Linear mappings and matrices.

In this section, we will see that linear mappings between finite-dimensional vector spaces have a matrix representation. The mechanism to achieve this matrix representation consists in using coordinates with respect to some basis of V and W.

More concretely, let $T: V \to W$ be a linear mapping, let $\mathcal{B}$ be a basis of V, and let C be a mapping of W. For any vector $\bar{x} \in V$, the image of $\bar{x}$ under T is $T\bar{x} \in W$. The coordinates of $\bar{x}$ and $T\bar{x}$ with respect to the corresponding bases are $[\bar{x}]_{\mathcal{B}}$ and $[T\bar{x}]_C$. We will see that there exists a matrix $M_T^{C,\mathcal{B}}$ such that

$$[T\bar{x}]_C = M_T^{C,B}\,[\bar{x}]_B. \qquad T(\bar{x})$$

The matrix $M_T^{C,B}$ is the matrix representation of T when we fix the bases B and C. This matrix converts the coordinates $[\bar{x}]_B$ into the coordinates $[T\bar{x}]_C$.

▶ The notation $M_T^{C,B}$ has been chosen to remark that the matrix represention of T depends of the bases B and C. In other words, it changes with the choice of bases.

Matrix associated to a linear mapping.

In this section, we will construct the matrix associated to the linear mapping $T: V \to W$.

We start by fixing a basis for V and W. Let $\mathcal{B}$ be a basis for V and let C be a basis for W. In particular, let

$$\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\}.$$

Then we can represent every $\bar{x} \in V$ as a linear combination of $\mathcal{B}$ using the coordinates $[\bar{x}]_{\mathcal{B}}$. That is,

$$\bar{x} = x_1\bar{b}_1 + x_2\bar{b}_2 + x_3\bar{b}_3 + \cdots + x_n\bar{b}_n \; ; \; [\bar{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Now, using the linearity of T, we have,

$$\begin{aligned} T\bar{x} &= T(x_1\bar{b}_1 + x_2\bar{b}_2 + x_3\bar{b}_3 + \cdots + x_n\bar{b}_n) \\ &= x_1T(\bar{b}_1) + x_2T(\bar{b}_2) + x_3T(\bar{b}_3) + \cdots + x_nT(\bar{b}_n). \end{aligned}$$

Since $T\bar{x}$ is a vector of W, we can take the coordinates of $T\bar{x}$ with respect to the basis C.

Therefore,

$$[T\bar{x}]_C = x_1[T(\bar{b}_1)]_C + x_2[T(\bar{b}_2)]_C + \cdots + x_n[T(\bar{b}_n)]_C$$

This last equation can be written as a matrix equation:

$$[T\bar{x}]_C = \big([T(\bar{b}_1)]_C \; [T(\bar{b}_2)]_C \ldots [T(\bar{b}_n)]_C\big)[\bar{x}]_{\mathcal{B}}.$$

Comparing with the equation $[T\bar{x}]_C = M_T^{C,\mathcal{B}}[\bar{x}]_{\mathcal{B}}$ at the beginning of the section, we see that

$$M_T^{C,\mathcal{B}} = \big([T(\bar{b}_1)]_C \; [T(\bar{b}_2)]_C \ldots [T(\bar{b}_n)]_C\big).$$

Matrix equation of a linear mapping

Let V and W be two vector spaces of dimensions n and m, respectively, and let $\mathcal{B}$ and C be bases of V and W, respectively. Given a linear mapping $T: V \to W$, the matrix equation of T with respect to $\mathcal{B}$ and C is

$$[\bar{y}]_C = [T\bar{x}]_C = M_T^{C,\mathcal{B}}[\bar{x}]_{\mathcal{B}} \iff T(\bar{x}) = \bar{y}$$

which, given the coordinates $[\bar{x}]_{\mathcal{B}}$ of a vector $\bar{x} \in V$ with respect to $\mathcal{B}$, computes the coordinates $[T\bar{x}]_{\mathcal{C}}$ of its image $T\bar{x}$ with respect to $\mathcal{C}$.

$M_T^{\mathcal{C},\mathcal{B}}$ is the matrix associated to T with respect to $\mathcal{B}$ and $\mathcal{C}$, that is: the matrix of size $m \times n$ given by

$$M_T^{\mathcal{C},\mathcal{B}} = \left([T(\bar{b}_1)]_{\mathcal{C}} \ [T(\bar{b}_2)]_{\mathcal{C}} \cdots [T(\bar{b}_n)]_{\mathcal{C}} \right).$$

- The notation $M_T^{\mathcal{C},\mathcal{B}}$ has been chosen so that, in the equation $[T\bar{x}]_{\mathcal{C}} = M_T^{\mathcal{C},\mathcal{B}} [\bar{x}]_{\mathcal{B}}$, everything indexed with $\mathcal{B}$ is collected on the right, and everything indexed with $\mathcal{C}$ is collected on the left.

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

$$Id(\bar{x}) = \bar{x} \qquad \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathcal{E}_2} \to \underbrace{\begin{bmatrix} y \\ x \end{bmatrix}}_{\mathcal{E}_2} \qquad T: \overset{\mathcal{B}}{V} \longrightarrow \overset{\mathcal{C}=\mathcal{B}}{V}$$

$$\begin{tikzcd} V \arrow[r, bend right, "Id."'] & V \end{tikzcd}$$

- A special case is when $V=W$. Nevertheless, $\mathcal{B}$ and $\mathcal{C}$ may not be the same bases. So a second special case when $V=W$ and $\mathcal{B}=\mathcal{C}$.

$$\underbrace{T}_{\mathcal{E}'}(\bar{x}) = \underbrace{A}_{M_T^{\mathcal{E}',\mathcal{E}}}\underbrace{\bar{x}}_{\mathcal{E}}$$

Example. Let $f:\mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping such that

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}.$$

We would like to write the matrix associated to f with respect to $\mathcal{E}_2$ and $\mathcal{E}_3$, the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Codomain ↘ ↙ Domain

$$M_f^{\mathcal{E}_3,\mathcal{E}_2} = \left(\left[f\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{E}_3} \quad \left[f\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{E}_3} \right)$$

$$= \left(\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right]_{\mathcal{E}_3} \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right]_{\mathcal{E}_3} \right)$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{pmatrix}$$

B

Example. Let $D: \mathbb{P}_3 \to \mathbb{P}_3$ be the linear mapping defined by $Dp(x) = p'(x)$. Consider the canonical basis $\mathcal{B} = \{1, x, x^2, x^3\}$. Write the matrix associated with D with respect to $\mathcal{B}$ (the second special case discussed above).

$$M_D^{B,B} = \left([D1]_B \;\; [Dx]_B \;\; [Dx^2]_B \;\; [Dx^3]_B \right)$$

$$= \left([0]_B \;\; [1]_B \;\; [2x]_B \;\; [3x^2]_B \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\bar{x} = 1 + 3x^2$ $\qquad$ $T\bar{x} = D(1+3x^2) = 6x$

$$[T\bar{x}]_B = M_D^{B,B}[\bar{x}]_B$$

derivative of $\bar{x} \in \mathbb{P}_3$

$$[T\vec{x}]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

$$[T\vec{x}]_B = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 6x$$

Example. Let $T: \mathbb{P}_3 \to \mathbb{P}_4$ be the linear mapping defined by $Tp(x) = xp(x)$. Consider the bases $\mathcal{B} = \{1, 1+x, x+x^2, x^2+x^3\}$ and $\mathcal{C} = \{1, x, x^2, x^3, x^4\}$ of $\mathbb{P}_3$ and $\mathbb{P}_4$, respectively.

Associated matrix: Injectivity and Surjectivity

In this section, we will relate some properties of a linear mapping with some properties of its associated matrix.

Consider the linear mapping $T: V \to W$ between to finite-dimensional vector spaces. Since we can associate a matrix with T, we will use the tools developed for matrices to study the mapping T.

For this, we need to fix a basis for V: $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\}$.

We will also need to fix a basis for W: C.

With these two bases, we can construct $M_T^{C,\mathcal{B}}$, the matrix associated with T, such that

$$[T\bar{x}]_C = M_T^{C,\mathcal{B}} [\bar{x}]_{\mathcal{B}}, \quad \forall \bar{x} \in V.$$

Recall that the matrix $M_T^{C,\mathcal{B}}$ is given by:

$$M_T^{C,\mathcal{B}} = \left([T(\bar{b}_1)]_C \ [T(\bar{b}_2)]_C \cdots [T(\bar{b}_n)]_C \right)$$

We have the following facts studied previously:

- Studying a set of vectors is equivalent to studying its coordinates.
- Since $\mathcal{B}$ is a linearly independent set, T is injective if and only if $\{T(\bar{b}_1), T(\bar{b}_2), \ldots, T(\bar{b}_n)\}$ is a linearly independent set.

- Since $\mathcal{B}$ is a generating system of V, T is surjective if and only if $\{T(\bar{b}_1), T(\bar{b}_2), \ldots, T(\bar{b}_n)\}$ is a generating system of W.

First, we will study the relationship between the injectivity of T and the pivot columns of $M_T^{\mathcal{C},\mathcal{B}}$. We know that T is injective if and only if $\{[T(\bar{b}_1)]_{\mathcal{C}}, [T(\bar{b}_2)]_{\mathcal{C}} \ldots, [T(\bar{b}_n)]_{\mathcal{C}}\}$ is linearly independent. But these vectors are the columns of $M_T^{\mathcal{C},\mathcal{B}}$. Therefore, we have

T is injective if and only if the columns of $M_T^{\mathcal{C},\mathcal{B}}$ are linearly independent, equivalently, all the columns are pivots.

- T is injective if and only if $[b]_C = M_T^{C,\mathcal{B}}[\bar{x}]_{\mathcal{B}}$, $\bar{b} \in \operatorname{Im} T$, has a unique solution.
- Clearly, the number of pivots in $M_T^{C,\mathcal{B}}$ is independent of the chosen bases $\mathcal{B}$ and C.
- We can relate the injectivity of $M_T^{C,\mathcal{B}}$ with the rank of $M_T^{C,\mathcal{B}}$

$$\operatorname{rank} M_T^{C,\mathcal{B}} = \#\text{pivots} \underset{\uparrow}{=} \#\text{columns of } M_T^{C,\mathcal{B}} = \dim V$$

by injectivity of T, all the columns are pivots.

Example. Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping such that

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}.$$

We have already found the associated matrix with respect to the canonical bases:

$$M_f^{\mathcal{E}_3,\mathcal{E}_2} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We can see that all the columns of $M_f^{\mathcal{E}_3,\mathcal{E}_2}$ are pivots (equivalently, $\operatorname{rank} M_f^{\mathcal{E}_3,\mathcal{E}_2} = \dim \mathbb{R}^2 = 2$), therefore f is injective.

Example. Let $D: \mathbb{P}_3 \to \mathbb{P}_3$ be the linear mapping defined by $Dp(x) = p'(x)$. As we have previously seen, the associated matrix with respect to the standard basis $\mathcal{B}$ is

$$M_D^{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix only has 3 pivots and one nonpivot column. Therefore D is not injective.
Equivalently, D is not injective because

$$\operatorname{rank} M_D^{\mathcal{B},\mathcal{B}} = 3 \neq \dim \mathbb{P}_3 = 4$$

Example. Let $T : \mathbb{P}_3 \to \mathbb{P}_4$ be the linear mapping defined by $Tp(x) = xp(x)$. Consider the bases $\mathcal{B} = \{1, 1+x, x+x^2, x^2+x^3\}$ and $\mathcal{C} = \{1, x, x^2, x^3, x^4\}$ of $\mathbb{P}_3$ and $\mathbb{P}_4$, respectively.

The associated matrix is

$$M_T^{\mathcal{C},\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case, every column of $M_T^{C,\mathcal{B}}$ is a pivot, and therefore T is injective. Equivalently, T is injective because

$$\text{rank } M_T^{C,\mathcal{B}} = 4 = \dim \mathbb{P}_3$$

Now, we will study the relationship between the surjectivity of T and the pivots of $M_T^{C,\mathcal{B}}$. We know that T is surjective if and only if $\{[T(\bar{b}_1)]_C, [T(\bar{b}_2)]_C, \ldots, [T(\bar{b}_n)]_C\}$ is a system of generators of $\mathbb{R}^m$, where $m = \dim W$. In other words, the matrix equation

$$[\bar{b}]_C = M_T^{C,\mathcal{B}} [\bar{x}]_{\mathcal{B}}, \quad \bar{b} \in W$$

should always have a solution for any $\bar{b} \in W$. For this to be true, each row of $M_T^{C,\mathcal{B}}$ must have a pivot in order to avoid contradictions of the form $0 = 1$.

$\boxed{\text{T is surjective if and only if the columns of } M_T^{\mathcal{C},\mathcal{B}} \text{ constitutes a system of generators of } \mathbb{R}^m,\ m=\dim W, \text{ equivalently, each row of } M_T^{\mathcal{C},\mathcal{B}} \text{ has a pivot.}}$

- T is surjective if and only if the equation $[\bar{b}]_{\mathcal{C}} = M_T^{\mathcal{C},\mathcal{B}} [\bar{x}]_{\mathcal{B}}$ always has a solution for any $\bar{b} \in W$.
- We can relate the surjectivity of T with the rank of $M_T^{\mathcal{C},\mathcal{B}}$

$$\operatorname{rank} M_T^{\mathcal{C},\mathcal{B}} = \#\text{pivots} \underset{\substack{\uparrow \\ \text{by the surjectivity} \\ \text{of } T\text{, each row has} \\ \text{a pivot.}}}{=} \#\text{rows of } M_T^{\mathcal{C},\mathcal{B}} = \dim W.$$

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping such that

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}.$$

We have already found the associated matrix with respect to the canonical bases:

$$M_f^{\mathcal{E}_3,\mathcal{E}_2} = \begin{pmatrix}1 & 2\\2 & 1\\3 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}.$$

We can see that not every row of $M_T^{\mathcal{E}_3,\mathcal{E}_2}$ has a pivot, therefore f is not surjective. Equivalently, f is not surjective because

$$\operatorname{rank} M_f^{\mathcal{E}_3,\mathcal{E}_2} = 2 \neq \dim \mathbb{R}^3 = 3$$

Example. Consider the linear mapping $T: \mathbb{R}^5 \to \mathbb{R}^3$ defined by $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{pmatrix} 1 & 4 & 7 & 5 & 11 \\ 2 & 5 & 8 & 7 & 13 \\ 3 & 6 & 10 & 9 & 16 \end{pmatrix}.$$

Clearly, the matrix associated to T with respect to the canonical bases is A.

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Since every row of A has a pivot, we have that T is surjective. Equivalently, T is surjective since

$$\operatorname{rank} A = 3 = \dim \mathbb{R}^3$$

- To finish this section, it must be remarked that T is bijective if and only if every row and every column of $M_T^{C,\mathcal{B}}$ has a pivot. For this, it is necessary that $M_T^{C,\mathcal{B}}$ is square and invertible.
- Moreover, a mapping can be both not injective and not surjective. An example of this is the mapping $D: \mathbb{P}_3 \to \mathbb{P}_3$ defined by $Dp(x) = p'(x)$.

Associated matrix and change of bases.

Let V and W be two vector spaces, and let $T: V \to W$ be a fixed, but arbitrary, linear mapping.

Recall that the matrix associated with T depends on the bases chosen for V and W. Nevertheless, the linear mapping should not depend on our choice of bases.
This implies that there is a relation between two matrices associated with T but with respect to distinct bases.

Indeed, these relations exist and constitute the center of attention of this section.

Recall that V and W are vector spaces and the linear mapping $T: V \longrightarrow W$ is fixed but arbitrary.

Let's choose two distinct bases for V: $\mathcal{B}$ and $\tilde{\mathcal{B}}$, and two distinct bases for W: $\mathcal{C}$ and $\tilde{\mathcal{C}}$.

Then, we have the following relations between coordinates:

$$[\bar{v}]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \tilde{\mathcal{B}}}{P} [\bar{v}]_{\tilde{\mathcal{B}}}, \quad \bar{v} \in V$$

$$[\bar{w}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \tilde{\mathcal{B}}}{P} [\bar{w}]_{\tilde{\mathcal{C}}}, \quad \bar{w} \in W,$$

where $\underset{\mathcal{B} \leftarrow \tilde{\mathcal{B}}}{P}$ and $\underset{\mathcal{C} \leftarrow \tilde{\mathcal{C}}}{P}$ are the corresponding matrices of change of bases.

Recall that $M_T^{\mathcal{C},\mathcal{B}}$ denotes the matrix associated with T with respect to $\mathcal{B}$ and $\mathcal{C}$, and $M_T^{\tilde{\mathcal{C}},\tilde{\mathcal{B}}}$ denotes the matrix associated with T with respect to $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$.

We can conveniently summarize the relation between these matrices with the following diagram

$$W \xleftarrow{\;T\;} V$$

$$\begin{tikzcd}
{[\bar{w}]_{\mathcal{C}} = M_T^{\mathcal{C},\mathcal{B}}} \arrow[d, "P_{\tilde{\mathcal{C}} \leftarrow \mathcal{C}}"'] & {[\bar{v}]_{\mathcal{B}}} \\
{[\bar{w}]_{\tilde{\mathcal{C}}} = M_T^{\tilde{\mathcal{C}},\tilde{\mathcal{B}}}} & {[\bar{v}]_{\tilde{\mathcal{B}}}} \arrow[u, "P_{\mathcal{B} \leftarrow \tilde{\mathcal{B}}}"']
\end{tikzcd}$$

$$\boxed{P_{\tilde{\mathcal{C}} \leftarrow \mathcal{C}} \, M_T^{\mathcal{C},\mathcal{B}} \, P_{\mathcal{B} \leftarrow \tilde{\mathcal{B}}}} = M_T^{\tilde{\mathcal{C}},\tilde{\mathcal{B}}}$$

Our goal is to find a relation between $M_T^{C,\mathcal{B}}$ and $M_T^{\tilde{C},\tilde{\mathcal{B}}}$. To do this, we proceed as follow:

Consider the matrix equation associated with T with respect to the bases $\mathcal{B}$ and C:

$$[T\bar{v}]_C = M_T^{C,\mathcal{B}} [\bar{v}]_{\mathcal{B}}, \quad \forall \bar{v} \in V.$$

Note that $[T\bar{v}]_C = \underset{C\leftarrow\tilde{C}}{P} [T\bar{v}]_{\tilde{C}}$ an $[\bar{v}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\tilde{\mathcal{B}}}{P} [\bar{v}]$. Substituting these relations in the above equation, we obtain

$$\underset{C\leftarrow\tilde{C}}{P} [T\bar{v}]_{\tilde{C}} = M_T^{C,\mathcal{B}} \underset{\mathcal{B}\leftarrow\tilde{\mathcal{B}}}{P} [\bar{v}]_{\tilde{\mathcal{B}}}$$

Lastly, multiplying both sides of this equation by $\underset{C\leftarrow\tilde{C}}{P}^{-1} = \underset{\tilde{C}\leftarrow C}{P}$, we obtain

$$[T\bar{v}]_{\tilde{C}} = \underset{\tilde{C}\leftarrow C}{P} \, M_T^{C,B} \, \underset{B\leftarrow \tilde{B}}{P} \, [\bar{v}]_{\tilde{B}}.$$

Comparing this last equation with $[T\bar{v}]_{\tilde{C}} = M_T^{\tilde{C},\tilde{B}} [\bar{v}]_{\tilde{B}}$, we deduce

$$\boxed{M_T^{\tilde{C},\tilde{B}} = \underset{\tilde{C}\leftarrow C}{P} \, M_T^{C,B} \, \underset{B\leftarrow \tilde{B}}{P}}$$

Change of bases formula for the matrix associated with a linear mapping T.

This formula is the relation between $M_T^{C,B}$ and $M_T^{\tilde{C},\tilde{B}}$ that we were looking for.

Observe how the notation can help us remember the formula above as follows:

On the left hand side of the formula we have $M_T^{\tilde{\mathcal{C}},\tilde{\mathcal{B}}}$ with superindeces $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{B}}$. On the right hand side, these same indices are on the exterior, while $\mathcal{B}$ and $\mathcal{C}$ appear on the interior. Moreover, notice that the subindex $\mathcal{B}$ y below the superindex $\mathcal{C}$, and similarly, the subindex $\mathcal{C}$ y below the superindex $\mathcal{C}$.

We can also use the diagram

$$
\begin{tikzcd}
W & V \arrow[l, "T"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[\bar{w}]_{\mathcal{C}}} \arrow[dd, "\underset{\tilde{\mathcal{C}}\leftarrow\mathcal{C}}{P}"'] & = & M_T^{\mathcal{C},\mathcal{B}} & {[\bar{v}]_{\mathcal{B}}} \\
 & & \boxed{\underset{\tilde{\mathcal{C}}\leftarrow\mathcal{C}}{P}\, M_T^{\mathcal{C},\mathcal{B}}\, \underset{\mathcal{B}\leftarrow\tilde{\mathcal{B}}}{P}} \arrow[d, equal] & \\
{[\bar{w}]_{\tilde{\mathcal{C}}}} & = & M_T^{\tilde{\mathcal{C}},\tilde{\mathcal{B}}} & {[\bar{v}]_{\tilde{\mathcal{B}}}} \arrow[uu, "\underset{\mathcal{B}\leftarrow\tilde{\mathcal{B}}}{P}"']
\end{tikzcd}
$$

to remember how to write the change of bases formula. Notice that the arrow connecting directly $\tilde{\mathcal{B}}$ and $\tilde{C}$ is labeled $M_T^{\tilde{C},\tilde{\mathcal{B}}}$. Also notice that $\tilde{\mathcal{B}}$ and $\tilde{C}$ can be connected alternatively via: $\tilde{\mathcal{B}} \longrightarrow \mathcal{B} \longrightarrow C \longrightarrow \tilde{C}$.

Writing (from right to left) the labels of this alternative path we get $\underset{\tilde{C}\leftarrow C}{P} M_T^{C,\mathcal{B}} \underset{\mathcal{B}\leftarrow\tilde{\mathcal{B}}}{P}$. Since the direct path and the alternative path connect the same two vertices $\tilde{\mathcal{B}}$ and $\tilde{C}$ we will say that they are equal: $M_T^{\tilde{C},\tilde{\mathcal{B}}} = \underset{\tilde{C}\leftarrow C}{P} \; M_T^{C,\mathcal{B}} \; \underset{\mathcal{B}\leftarrow\tilde{\mathcal{B}}}{P}$, obtaining the desired change of bases formula.

Example. Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping such that

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}.$$

We have already found the associated matrix with respect to the canonical bases:

$$M_f^{\mathcal{E}_3, \mathcal{E}_2} = \begin{pmatrix}1 & 2\\2 & 1\\3 & 0\end{pmatrix}$$

Now consider the following bases for $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

We will compute the matrix associated with f with respect to $\mathcal{B}$ and $\mathcal{C}$.

Example. Consider the linear mapping $T:\mathbb{R}^5 \to \mathbb{R}^3$ defined by $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{pmatrix} 1 & 4 & 7 & 5 & 11 \\ 2 & 5 & 8 & 7 & 13 \\ 3 & 6 & 10 & 9 & 16 \end{pmatrix}.$$

Clearly, the matrix associated to T with respect to the canonical bases $\mathcal{E}_5$ and $\mathcal{E}_3$ is $M_T^{\mathcal{E}_3,\mathcal{E}_5} = A$.

Consider the bases of $\mathbb{R}^5$ and $\mathbb{R}^3$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix} \right\}$$

The matrix associated with T with respect to $\mathcal{B}$ and $\mathcal{C}$ is given by